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FUNCTIONAL INEQUALITIES IN PARANORMED SPACES

Sung Jin Lee*, Choonkil Park**, and Jung Rye Lee***

Abstract. In this paper, we investigate additive functional inequalities in paranormed spaces. Furthermore, we prove the Hyers-Ulam stability of additive functional inequalities in paranormed spaces.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [3] and Steinhaus [26] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [5, 14, 16, 17, 25]). This notion was defined in normed spaces by Kolk [15].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [28] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \le P(x) + P(y)$ (triangle inequality)
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have
(5) $P(x) = 0$ implies $x = 0$.

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Correspondence should be addressed to Jung Rye Lee, jrlee@daejin.ac.kr.
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A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [27] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [22] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] following the same approach as in Rassias [21], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Rassias and Šemrl [23] that one cannot prove a Rassias' type theorem when $p = 1$ (cf. the books of Czerwik [2], Hyers, Isac and Rassias [11]).

In 1982, J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias' theorem [21] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$. Găvruta [7] provided a further generalization of Th.M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings (see [12, 13, 18]).

Throughout this paper, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

In [8], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [24]. Fechner [4] and Gilányi [9] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park, Cho and Han [19] proved the Hyers-Ulam stability of the following functional inequalities

$$\|f(x) + f(y) + f(z)\| \leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \tag{1.2}$$

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\|, \tag{1.3}$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\|. \tag{1.4}$$

In Section 2, we prove the Hyers-Ulam stability of the functional inequality (1.2) in paranormed spaces. In Section 3, we prove the Hyers-Ulam stability of the functional inequality (1.3) in paranormed spaces. In Section 4, we prove the Hyers-Ulam stability of the functional inequality (1.4) in paranormed spaces.

2. Stability of a functional inequality associated with a 3-variable Jensen additive functional equation

We prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type 3-variable Jensen additive functional equation.

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

PROPOSITION 2.1. ([19, Proposition 2.1]) *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \le \left\| 2f\left(\frac{x+y+z}{2}\right)\right\|$$

for all $x, y, z \in X$. Then f is Cauchy additive.

THEOREM 2.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x) + f(y) + f(z)\| \\ &\le \left\| 2f\left(\frac{x+y+z}{2}\right)\right\| + P(x)^r + P(y)^r + P(z)^r \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{4 + 2^{r+2} + 2^{2r}}{4 - 4^r} P(x)^r = \frac{2 + 2^r}{2 - 2^r} P(x)^r \tag{2.2}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.1), we get $\|3f(0)\| \le \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ and $z = -2x$ in (2.1), we get

$$\|2f(x) + f(-2x)\| \le (2 + 2^r)P(x)^r$$

and so

$$\|2f(-2x) + f(4x)\| \le (2 + 2^r)2^r P(x)^r$$

for all $x \in X$. Thus

$$\|4f(x) - f(4x)\| \leq (4 + 2^{r+2} + 2^{2r})P(x)^r$$

and so

$$\left\| f(x) - \frac{1}{4} f(4x) \right\| \leq \frac{4 + 2^{r+2} + 2^{2r}}{4} P(x)^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{4^l} f(4^l x) - \frac{1}{4^m} f(4^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f(4^j x) - \frac{1}{4^{j+1}} f(4^{j+1} x) \right\| \\ &\leq \frac{4 + 2^{r+2} + 2^{2r}}{4} \sum_{j=l}^{m-1} \frac{4^{rj}}{4^j} P(x)^r \qquad (2.3) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.3) that the sequence $\{\frac{1}{4^n} f(4^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(4^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n \to \infty} \frac{1}{4^n} f(4^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.3), we get (2.2).

It follows from (2.1) that

$$\begin{aligned} &\|h(x) + h(y) + h(z)\| = \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n x) + f(4^n y) + f(4^n z)\| \\ &\leq \lim_{n \to \infty} \frac{1}{4^n} \left\| 2f\left(4^n \frac{x+y+z}{2}\right) \right\| + \lim_{n \to \infty} \frac{4^{nr}}{4^n} (P(x)^r + P(y)^r + P(z)^r) \\ &= \left\| 2h\left(\frac{x+y+z}{2}\right) \right\| \end{aligned}$$

for all $x, y, z \in X$. So

$$\|h(x) + h(y) + h(z)\| \leq \left\| 2h\left(\frac{x+y+z}{2}\right) \right\|$$

for all $x, y, z \in X$. By Proposition 2.1, the mapping $h : X \to Y$ is Cauchy additive.

Now, let $T : X \to Y$ be another Cauchy additive mapping satisfying (2.2). Then we have

$$\begin{aligned}\|h(x) - T(x)\| &= \frac{1}{4^n}\left\|h\left(4^n x\right) - T\left(4^n x\right)\right\| \\ &\leq \frac{1}{4^n}\left(\left\|h\left(4^n x\right) - f\left(4^n x\right)\right\| + \left\|T\left(4^n x\right) - f\left(4^n x\right)\right\|\right) \\ &\leq \frac{2(4 + 2^{r+2} + 2^{2r})4^{nr}}{(4 - 4^r)4^n} P(x)^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique Cauchy additive mapping satisfying (2.2). $\square$

THEOREM 2.3. *Let r be a positive real number with $r < \frac{1}{3}$, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(x) + f(y) + f(z)\| \\ &\leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\| + P(x)^r \cdot P(y)^r \cdot P(z)^r\end{aligned} \tag{2.4}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2^{r+1} + 2^{4r}}{4 - 4^{3r}} P(x)^{3r} \tag{2.5}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.4), we get $\|3f(0)\| \leq \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ and $z = -2x$ in (2.4), we get

$$\|2f(x) + f(-2x)\| \leq 2^r P(x)^{3r}$$

and so

$$\|2f(-2x) + f(4x)\| \leq 2^{4r} P(x)^{3r}$$

for all $x \in X$. Thus

$$\|4f(x) - f(4x)\| \leq (2^{r+1} + 2^{4r}) P(x)^{3r}$$

and so

$$\left\|f(x) - \frac{1}{4} f(4x)\right\| \leq \frac{2^{r+2} + 2^{4r}}{4} P(x)^{3r}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(4^l x) - \frac{1}{4^m}f(4^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f(4^j x) - \frac{1}{4^{j+1}}f(4^{j+1}x)\right\| \\ &\leq \frac{2^{r+1}+2^{4r}}{4}\sum_{j=l}^{m-1}\frac{4^{3rj}}{4^j}P(x)^{3r} \qquad (2.6)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{\frac{1}{4^n}f(4^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(4^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(4^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.5).

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Stability of a functional inequality associated with a 3-variable Cauchy additive functional equation

We prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type 3-variable Cauchy additive functional equation.

PROPOSITION 3.1. ([19, Proposition 2.2]) *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|$$

for all $x, y, z \in X$. Then f is Cauchy additive.

THEOREM 3.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(x) + f(y) + f(z)\| \\ &\leq \|f(x + y + z)\| + P(x)^r + P(y)^r + P(z)^r \qquad (3.1)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2 + 2^r}{2 - 2^r}P(x)^r$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.1), we get $\|3f(0)\| \leq \|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ and $z = -2x$ in (3.1), we get

$$\|2f(x) + f(-2x)\| \leq (2 + 2^r)P(x)^r$$

and so

$$\|2f(-2x) + f(4x)\| \leq (2 + 2^r)2^r P(x)^r$$

for all $x \in X$. Thus

$$\|4f(x) - f(4x)\| \leq (4 + 2^{r+2} + 2^{2r})P(x)^r$$

and so

$$\left\|f(x) - \frac{1}{4}f(4x)\right\| \leq \frac{4 + 2^{r+2} + 2^{2r}}{4}P(x)^r$$

for all $x \in X$.

The rest of the proof is the same as in the proof of Theorem 2.2. □

THEOREM 3.3. *Let r be a positive real number with $r < \frac{1}{3}$, and let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + P(x)^r \cdot P(y)^r \cdot P(z)^r$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2^{r+1} + 2^{4r}}{4 - 4^{3r}}P(x)^{3r}$$

for all $x \in X$.

Proof. The proof is similar to the proofs of Theorems 2.2 and 2.3. □

4. Stability of a functional inequality associated with the Cauchy-Jensen functional equation

We prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type Cauchy-Jensen functional equation.

PROPOSITION 4.1. ([19, Proposition 2.3]) *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\|$$

for all $x, y, z \in X$. Then f is Cauchy additive.

THEOREM 4.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| + P(x)^r + P(y)^r + P(z)^r \tag{4.1}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2 + 3 \cdot 2^r + 2^{2r}}{4 - 4^r} P(x)^r = \frac{1 + 2^r}{2 - 2^r} P(x)^r$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (4.1), we get $\|4f(0)\| \leq \|2f(0)\|$. So $f(0) = 0$.

Replacing x by $-2x$ and letting $y = 0$ and $z = x$ in (4.1), we get

$$\|f(-2x) + 2f(x)\| \leq (1 + 2^r)P(x)^r$$

and so

$$\|2f(-2x) + f(4x)\| \leq (1 + 2^r)2^r P(x)^r$$

for all $x \in X$. Thus

$$\|4f(x) - f(4x)\| \leq (2 + 3 \cdot 2^r + 2^{2r})P(x)^r$$

and so

$$\left\|f(x) - \frac{1}{4}f(4x)\right\| \leq \frac{2 + 3 \cdot 2^r + 2^{2r}}{4} P(x)^r$$

for all $x \in X$.

The rest of the proof is the same as in the proof of Theorem 2.2. □

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Department of Mathematics
Daejin University
Kyeonggi 487-711, Republic of Korea
E-mail: hyper@daejin.ac.kr

**
Department of Mathematics
Hanyang University
Seoul 133-791, Republic of Korea
E-mail: baak@hanyang.ac.kr

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Department of Mathematics
Daejin University
Kyeonggi 487-711, Republic of Korea
E-mail: jrlee@daejin.ac.kr